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On Strongly Closed Subgraphs of Highly Regular Graphs

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Abstract

A geodetically closed induced subgraph Δ of a graph Γ is defined to be strongly closed if $\Gamma_i(\alpha) \cap \Gamma_1(\beta)$ stays in Δ for every i and α, $\beta \in \Delta$ with $\partial(\alpha, \beta) = i$. We study the existence conditions of strongly closed subgraphs in highly regular graphs such as distance-regular graphs or distance-biregular graphs.

1 Introduction

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph. For a subset $\Delta \subset V(\Gamma)$, we identify Δ with the induced subgraph on Δ. In particular, $\Gamma = V(\Gamma)$.

For two vertices α, β in Γ, let $\partial_\Gamma(\alpha, \beta)$ denote the distance between α and β in Γ, i.e., the length of a shortest path connecting α and β in Γ. We also write $\partial(\alpha, \beta)$, when no confusion occurs. Let

$$\Gamma_i(\alpha) = \{\beta \in \Gamma | \partial(\alpha, \beta) = i\} \text{ and } \Gamma(\alpha) = \Gamma_1(\alpha).$$

For vertices α, β in Γ with $\partial(\alpha, \beta) = i$, let

$$\begin{aligned}
C(\alpha, \beta) &= C_i(\alpha, \beta) = \Gamma_{i-1}(\alpha) \cap \Gamma(\beta), \\
A(\alpha, \beta) &= A_i(\alpha, \beta) = \Gamma_i(\alpha) \cap \Gamma(\beta), \\
B(\alpha, \beta) &= B_i(\alpha, \beta) = \Gamma_{i+1}(\alpha) \cap \Gamma(\beta), \text{ and} \\
G(\alpha, \beta) &= \bigcup_{j=0}^{i} \Gamma_j(\alpha) \cap \Gamma_{i-j}(\beta) \\
&= \{\gamma \in \Gamma | \partial(\alpha, \gamma) + \partial(\gamma, \beta) = \partial(\alpha, \beta)\}.
\end{aligned}$$

$G(\alpha, \beta)$ is the set of vertices which lie on a geodesic between α and β. For the cardinalities, we use lower case letters, i.e.,

$$c_i(\alpha, \beta) = |C_i(\alpha, \beta)|, \ a_i(\alpha, \beta) = |A_i(\alpha, \beta)|, \text{ and } b_i(\alpha, \beta) = |B_i(\alpha, \beta)|.$$

We also write $c_i(\alpha)$ [resp. $a_i(\alpha)$, $b_i(\alpha)$] if the number $c_i(\alpha, \beta)$ [resp. $a_i(\alpha, \beta)$, $b_i(\alpha, \beta)$] does not depend on the choice of β under the condition $\partial(\alpha, \beta) = i$, and c_i [resp. a_i, b_i] if the number $c_i(\alpha, \beta)$ [resp. $a_i(\alpha, \beta)$, $b_i(\alpha, \beta)$] does not depend on the choices of α and β

under the condition $\partial(\alpha,\beta) = i$. In these cases we say for example that $c_i(\alpha)$ exists or c_i exists.

A connected graph Γ is said to be *distance-regular* if c_i, a_i, b_i exist for all i.

A connected bipartite graph Γ with a bipartition $P \cup L$ is said to be *distance-biregular* if $c_i(\alpha)$, $b_i(\alpha)$ exist for all i and these numbers depend only on the part α belongs to.

For convienience, if $\Gamma = P \cup L$ is a bipartite graph, we also use notations like c_i^P, b_i^P, c_i^L, b_i^L, when the corresponding numbers depend only on the part the base point belongs to.

A subset Δ of a graph Γ is said to be *C_i-closed* [resp. *A_i-closed*] if $C_i(\alpha,\beta) \subset \Delta$ [resp. $A_i(\alpha,\beta) \subset \Delta$] for every pair of vertices α, β in Δ with $\partial_\Gamma(\alpha,\beta) = i$.

A subset Δ of Γ is said to be *geodetically closed* if $C(\alpha,\beta) \subset \Delta$ for every pair of vertices α, β in Δ, i.e., Δ is C_i-closed for every i. In this case, we have $\partial_\Gamma(\alpha,\beta) = \partial_\Delta(\alpha,\beta)$ for all α, $\beta \in \Delta$. It is clear that Δ is geodetically closed if and only if $G(\alpha,\beta) \subset \Delta$ for every pair of vertices α, β in Δ.

A subset Δ of Γ is said to be *strongly closed* if $C(\alpha,\beta) \subset \Delta$ and $A(\alpha,\beta) \subset \Delta$ for every pair of vertices α, β in Δ, i.e., Δ is both C_i-closed and A_i-closed for every i.

We call the induced subgraph on Δ a geodetically [resp. strongly] closed subgraph when Δ is a geodetically [resp. strongly] closed subset.

By definition, every strongly closed subgraph is geodetically closed, in particular connected if Γ is connected. When Γ is bipartite, every geodetically closed subgraph is strongly closed and we do not need to distinguish these notions.

In most known distance-regular graphs, there are many nontrivial geodetically closed subgraphs and in many cases they are even strongly closed. In some cases we can guarantee the existence of strongly [or geodetically] closed subgraphs if we know a part of the parameters c_i, a_i. See [6, 18, 19, 21, 24], and [5, Section 4.3]. We believe that the investigation of strongly [or geodetically] closed subgraphs is a key in the study of distance-regular graphs.

The first question is the following:

> *Is a strongly closed subgraph Δ of a distance-regular graph Γ always distance-regular?*

By definition, the answer is 'yes' if Δ is regular. On the contrary, we can find counter examples easily. For example, if the girth of Γ is large, we can construct a strongly closed subgraph isomorphic to a tree.

Are there any other types of non-regular strongly closed subgraphs of distance-regular graphs? Theorem 1.1 gives a solution to this problem.

We need a few more definitions to state the theorem.

Let $l(c,a,b) = |\{i|(c_i,a_i,b_i) = (c,a,b)\}|$ and $r(\Gamma) = l(c_1,a_1,b_1)$.

Let $d(\Gamma) = \max\{\partial(\alpha,\beta)|\alpha,\beta \in \Gamma\}$, and $k(\alpha) = |\Gamma(\alpha)| = b_0(\alpha,\alpha)$.

Let K_{k+1} denote the complete graph of valency k, and M_k denote a Moore graph of valency k, which is known to be of diameter 2 and $k \in \{2, 3, 7, 57\}$.

For a graph Γ, ${}^t\Gamma$ denotes a subdivision graph obtained by replacing each edge by a path of length t.

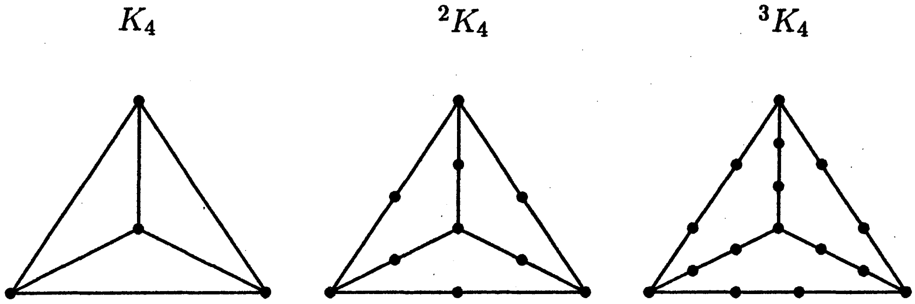

Figure 1.

Theorem 1.1 *Let Δ be a strongly closed subgraph of a distance-regular graph Γ. Then one of the following holds.*

(i) *Δ is a distance-regular graph,*

(ii) $2 \leq d(\Delta) \leq r(\Gamma)$,

(iii) *Δ is a distance-biregular graph with $c_{2i-1} = c_{2i}$ for all i with $2i \leq d(\Delta)$. In particular, $r(\Gamma) \equiv d(\Delta) \equiv 0 \pmod 2$; or*

(iv) *Δ is a subdivision graph of a complete graph or a Moore graph obtained by replacing each edge by a path of length three, i.e., $\Delta \simeq {}^3K_{l+1}$ or ${}^3M_l$. In particular, $d(\Delta) = r(\Gamma) + 2 = 5$ or 8, and $a_1 = 0$, $c_{r+1} = c_{r+2} = a_{r+1} = a_{r+2} = 1$, where $r = r(\Gamma)$.*

In particular, $(c_{m-1}, a_{m-1}, b_{m-1}) = (c_m, a_m, b_m)$ with $d(\Delta) = m$, except the case (i).

For the corresponding result when Γ is a distance-biregular graph, see the following section.

It follows easily from Theorem 1.1 that if $c_2 \neq 1$, then every strongly closed subgraph in a distance-regular graph is distance-regular. Using this fact, one can prove the following theorem without difficulty, and it is useful when one wants to characterize a distance-regular graph Γ by the structure of its antipode $\Gamma_d(\alpha)$.

Theorem 1.2 **([28])** *Let Γ be a distance-regular graph of diameter $d = d(\Gamma)$. If $\Gamma_d(\alpha)$ is strongly closed for some $\alpha \in \Gamma$, then $\Gamma_d(\beta)$ is a clique for every vertex $\beta \in \Gamma$.*

The second question is the following:

Can we find parametrical conditions for distance-regular graphs to have strongly closed subgraphs?

In this paper, we shall discuss this problem for the cases (iii) and (iv). Note that if $2 \leq m \leq r(\Gamma)$, then we can find a strongly closed subgraph Δ in Γ of diameter m which is, roughly speaking, isomorphic to a graph obtained by replacing each edge of a tree by a clique.

Case (iii) is treated in Sections 3 and 4. In this case we have $a_i = 0$ for $i \leq d(\Delta)$. Though we discuss in full generality, it seems more natural to state the results on bipartite graphs. The first result in this case is an improvement of a result of Ray-Chaudhuri and Sprague on pseudo-projective incidence systems.

Let $q = p^e$ be a prime power and V be a d-dimensional vector space over $GF(q)$ when $q \neq 1$, and a d-element set when $q = 1$. Let $\begin{bmatrix} V \\ i \end{bmatrix}_q$ denote the collection of i-dimensional subspaces of V when $q \neq 1$, and the collection of i-subsets when $q = 1$.

Let $J_q(d, s, s-1)$ denote a bipartite graph with a bipartition

$$\begin{bmatrix} V \\ s-1 \end{bmatrix}_q \cup \begin{bmatrix} V \\ s \end{bmatrix}_q,$$

where $x \in \begin{bmatrix} V \\ s-1 \end{bmatrix}_q$, $l \in \begin{bmatrix} V \\ s \end{bmatrix}_q$ is adjacent if and only if $x \subset l$. $J_q(d, s, s-1)$ is a distance-biregular graph and is called an (s, q, d)-projective incidence structure in [24].

Throughout this paper, we make a convention that $(q^m - 1)/(q-1) = m$, when $q = 1$.

Theorem 1.3 *Let Γ be a connected bipartite graph of diameter at least five with a bipartition $P \cup L$. Suppose $c_2(x) = 1, c_3(x) = c_4(x) = q+1$ for every $x \in P$, where q is a fixed positive integer. Then Γ is a biregular graph of valencies $k^P = k(x)$, and $k^L = k(l)$, where $x \in P$, $l \in L$. If $c_5(x)$ exists for every $x \in P$ and does not depend on the choice of $x \in P$, then one of the following holds.*

(i) $\Gamma \simeq J_q(d, s, s-1)$, *where* $k^L = (q^s - 1)/(q-1)$, $k^P = (q^{d-s+1} - 1)/(q-1)$, *or*

(ii) $d(\Gamma) \leq 7$, $q \neq 1$, k^P, $k^L \leq 3q - 1$.

In particular, q is a power of a prime if $k^P \geq 3q$ or $k^L \geq 3q$.

In [20] Koolen conjectured that under the hypothesis slightly stronger than that of Theorem 1.3, (i) or $d(\Gamma) \leq 4$ holds. Hence Theorem 1.3 gives an affirmative (but not complete) solution to the conjecture. For the detailed information on the case (ii), see Section 3.

Ray-Chaudhuri and Sprague obtained only the case (i) under an additional hypothesis $q^2 + q + 1 \leq k^L$. So in this paper we shall treat the case when the valency is not so large compared with c_3^P, using a result of Terwilliger in [30]. In any case, as we can guess from the conclusion, one of the keys is to show that every pair of vertices of distance four determines a geodetically closed (hence, strongly closed but not regular) subgraph of diameter four assuming that the valency k^L is not so small. See Section 3.

Let Γ be a distance-biregular graph with a bipartition $P \cup L$. Assume r is even and

$$c_r^P = 1 < c_{r+1}^P = c_{r+2}^P.$$

This is one of the typical cases corresponding to Theorem 1.1.(iii). By Theorem 1.3, if $r = 2$ and $d(\Gamma) \geq 8$, then Γ contains a strongly closed subgraph, which is distance-biregular of diameter four. It seems unlikely to have $r > 4$ and $r = 4$ is rare. We do not have a proof, but we can prove that $r \leq 4$ if Γ contains a strongly closed subgraph of diameter $r + 2$. See Section 3. In Section 4, we treat the case $c_{r+1}^P = 2$ with $r = 4$ and prove the following.

Theorem 1.4 *Let Γ be a connected bipartite graph with a bipartition $P \cup L$. Suppose $c_2(x) = c_3(x) = c_4(x) = 1$, $c_5(x) = c_6(x) = 2$ for every $x \in P$. Then Γ is a biregular graph of valencies k^P and k^L. If α, β be vertices in Γ with $\partial(\alpha, \beta) = 5$, then there is a strongly closed subgraph Δ containing α and β isomorphic to ${}^2M_{k^P}$. In particular, $k^P \in \{2, 3, 7, 57\}$, if $d(\Gamma) \geq 5$.*

We can show under the hypothesis in Theorem 1.4 that c_i^L exists for $i = 1, 2, 3, 4, 5, 6$, $c_1^L = \cdots c_4^L = 1$ and $c_5^L = c_6^L = 2$. Hence Theorem 1.4 implies that $k^L \in \{2, 3, 7, 57\}$ as well. When $k^P = 2$ or $k^L = 2$, Γ itself is a subdivision graph of a Moore graph isomorphic to ${}^2M_k$ for some k. When $k^P = k^L = 3$, Foster graph is an example. We do not know any other examples. It may be possible to classify Γ satisfying the condition of Theorem 1.4.

Case(iv) in Theorem 1.1 is treated in Section 5, under an additional condition $c_{r+3} = 1$.

Theorem 1.5 *Let Γ be a distance-regular graph of valency $k > 2$ satisfying the following.*

$$\begin{aligned} (c_r, a_r, b_r) &= (1, 0, k-1), \\ (c_{r+1}, a_{r+1}, b_{r+1}) &= (c_{r+2}, a_{r+2}, b_{r+2}) = (1, 1, k-2), \end{aligned}$$

$r \geq 1$ and $c_{r+3} = 1$. Then $r \equiv 0 \pmod 3$, and the following holds.

(1) If $r = 3$, then for every α, $\beta \in \Gamma$ with $\partial(\alpha, \beta) = 3$, there is a strongly closed subgraph Δ containing α, β isomorphic to ${}^3K_{k+1}$.

(2) If $r = 6$, then for every α, $\beta \in \Gamma$ with $\partial(\alpha, \beta) = 6$, there is a strongly closed subgraph Δ containing α, β isomorphic to ${}^3M_k$. In particular $k \in \{3, 7, 57\}$.

The first part $r \equiv 0 \pmod 3$ is due to Boshier-Nomura [4]. It is known that if $l(1, 0, k-1) = r \geq 1$, then $l(1, 1, k-2) \leq 3$ and if $l(1, 1, k-2) = 3$, then $c_{r+4} > 1$ [4, 13].

It is worth mentioning that both results Theorem 1.4 and 1.5 are related to circuit chasing technique. See [26] for a result related to Theorem 1.4.

We use intersection diagrams as our tools. We refer those who are not familiar with them to [4, 13, 14, 16, 23, 25, 26] and [5, Section 5.10] for example.

For subsets A, B of Γ let $e(A, B)$ denote the number of edges between A and B, and $e(x, A) = e(\{x\}, A)$.

$\Gamma^{(i)}$ will denote the distance-i-graph on Γ, i.e., the graph defined on the vertex set $V(\Gamma)$ of Γ such that α and β are adjacent if and only if $\partial_\Gamma(\alpha, \beta) = i$.

We write $\alpha \sim \beta$ when $\alpha \in \Gamma(\beta)$.

2 Strongly Closed Subgraphs

We shall prove Theorem 1.1 and related results in this section. The key of the proof is the determination of graphs such that c_i's and a_i's exist. Problems in similar settings are discussed in [12, 30, 20].

Proposition 2.1 *Let Γ be a connected bipartite graph with a bipartition $P \cup L$. Suppose c_i^P exists for $i = 1, \ldots, m$ with $m \leq d(\Gamma)$. If $c_1^P = \cdots = c_r^P = 1 < c_{r+1}^P$, with $r + 1 \leq m$, then the following hold.*

(1) *If $c_i^P = c_{i-1}^L$ for some $i \leq m$, then c_i^L exists and $c_{i-1}^P = c_i^L$. In particular, $c_1^L, \ldots, c_r^L$ exist and $c_1^L = \cdots = c_r^L = 1$.*

(2) *If $c_1^L, \ldots, c_{2i}^L$ exist and $2i + 1 \leq m$, then c_{2i+1}^L exists and $c_{2j}^P c_{2j+1}^P = c_{2j}^L c_{2j+1}^L$ for all $j \leq i$.*

(3) *If r is even, then Γ is biregular of valencies b_0^P and b_0^L. Moreover c_{r+1}^L exists and $c_{r+1}^P = c_{r+1}^L$.*

(4) *If r is odd, and c_{r+1}^L exists, then Γ is biregular of valencies b_0^P and b_0^L. Moreover,*

$$(c_{r+1}^P - 1)(b_0^L - 1) = (c_{r+1}^L - 1)(b_0^P - 1).$$

(5) *Suppose Γ is biregular of valencies $k^P = b_0^P$ and $k^L = b_0^L$. Then $|P|k^P = |L|k^L$. Moreover, if $c_1^L, \ldots, c_{2i}^L$ exist with $2i \leq m$, then b_s^P, b_t^L exist for $s \leq m$, $t \leq 2i$ and $b_{2j-1}^P b_{2j}^P = b_{2j-1}^L b_{2j}^L$, for all $j \leq i$.*

We can obtain the following theorem as a direct corollary by applying Proposition 2.1 to Δ.

Theorem 2.2 *Let Γ be a connected bipartite graph with a bipartition $P \cup L$. Suppose c_i^P, c_i^L exist for $i = 1, \ldots, m$. Let $c_1^P = \cdots = c_r^P = 1 < c_{r+1}^P$ with $r + 1 \leq m$. If Δ is a geodetically closed subgraph of Γ of diameter m, then Δ is a distance-biregular graph.*

Remark. For a diatance-biregular graph $\Gamma = P \cup L$, let $d^P = \max\{\partial(x,\alpha) | \alpha \in \Gamma\}$, where $x \in P$, and $d^L = \max\{\partial(l,\alpha) | \alpha \in \Gamma\}$, where $l \in L$. In Theorem 2.2, if $d^{P\cap\Delta} \geq d^{L\cap\Delta}$, then $k^{P\cap\Delta} = c^P_m$. But we cannot determine the other valency when $d^{P\cap\Delta} > d^{L\cap\Delta}$.

Proposition 2.3 *Let Γ be a connected graph. Suppose c_i exists for $i = 1,\dots,m$ with $m \leq d(\Gamma)$. Suppose $c_1 = \cdots = c_r = 1$, $a_1,\dots,a_r$ exist and $a_1 = \cdots = a_r$ and either $c_{r+1} > 1$ or $c_{r+1} = 1$ and a_{r+1} exists with $a_{r+1} \neq a_1$, where $2 \leq r+1 \leq m$. Then one of the following holds.*

(i) *Γ is regular.*

(ii) *Γ is a bipartite biregular graph such that $r \equiv 0 \pmod 2$ and $c_{2i-1} = c_{2i}$ for all i with $2i \leq m$.*

(iii) *$\Gamma \simeq {}^3K_{k+1}$ or ${}^3M_k$, where k is the largest valency of a vertex in Γ. In particular, $r = 3$ or 6.*

Lemma 2.4 *Let Γ be a connected graph of diameter $d = d(\Gamma)$. Suppose c_d, c_{d-1}, a_d, a_{d-1} exist. Then Γ is regular of valency $c_d + a_d$ if and only if $(c_{d-1}, a_{d-1}) \neq (c_d, a_d)$.*

Lemma 2.5 *Let Γ be a distance-regular graph of diameter $d = d(\Gamma)$ and $m < d$. Suppose Γ has a strongly closed subgraph of diameter m containing α and β for every pair of vertices α, β with $\partial(\alpha,\beta) = m$. Then for all γ, $\delta \in \Gamma$ with $\partial(\gamma,\delta) \leq m+1$, $C(\gamma,\delta)$ is a coclique.*

Now we prove Theorem 1.1 under weaker conditions.

Theorem 2.6 *Let Γ be a connected graph of diameter $d = d(\Gamma)$. Suppose c_i's and a_i's exist for all $i = 1,\dots,m$, where $m \leq d$. Let*

$$r = r(\Gamma) = \max\{i | (c_1,a_1) = (c_2,a_2) = \cdots = (c_i,a_i)\}.$$

If Γ contains a strongly closed subgraph Δ of diameter m, then one of the following holds.

(i) *Δ is a distance-regular graph,*

(ii) *$2 \leq m \leq r$,*

(iii) *Δ is a distance-biregular graph and that $r \equiv m \equiv 0 \pmod 2$ and $c_{2i-1} = c_{2i}$ for all i with $2i \leq m$, or*

(iv) *$\Delta \simeq {}^3K_{l+1}$ or ${}^3M_l$ and $m = r+2 = 5$ or 8, $a_1 = \cdots = a_r = 0$, $c_1 = \cdots = c_{r+2} = a_{r+1} = a_{r+2} = 1$.*

Proof. Since Δ is a strongly closed subgraph of Γ, we can apply Proposition 2.3 to the subgraph Δ. If $r \geq m$, then (i) or (ii) holds.

Assume $r+1 \leq m$. Then Δ is one of the types in Proposition 2.3. If Δ is regular, then Δ is distance-regular as c_i's and a_i's exist for $i \leq d(\Delta) = m$. Suppose Δ is not regular. Since Δ is strongly closed, $k(\alpha) = k(\beta)$ if α, $\beta \in \Delta$ and $\partial(\alpha, \beta) = m$. So if Δ is a bipartite biregular graph, Δ is distance-biregular and $m \equiv 0 \pmod 2$. Hence we have (iii). Suppose $\Delta \simeq {}^3K_{l+1}$ or ${}^3M_l$. Then $r = 3$ or 6 and $m = r+2$, $c_1 = \cdots = c_m = 1$, $a_1 = \cdots = a_r = 0$, $a_{r+1} = a_{r+2} = 1$ easily follow from the structure of Δ.

Lemma 2.7 *Let Γ be a distance-biregular graph with a bipartition $P \cup L$. Suppose k^P, $k^L \geq 2$. Let $d = d(\Gamma)$,*

$$d^P = \max\{\partial(x,\alpha) | x \in P,\ \alpha \in \Gamma\}, \quad d^L = \max\{\partial(l,\alpha) | l \in L,\ \alpha \in \Gamma\},$$

and $r(\Gamma) = \max\{i | c_i^P = 1\}$. Then $r(\Gamma) = \max\{i | c_i^L = 1\}$ and the following are equivalent.

(i) $r(\Gamma) + 2 = d^P + 1 = d^L = d$.

(ii) $d = d^L = r(\Gamma) + 2$, $c_{d-1}^L = c_d^L$ *with d even.*

In this case Γ is a Moore geometry and $d = 4$ or 6. If $d = 4$, Γ is nothing but a nonsymmetric 2-$(|P|, k^L, 1)$ design. If $d = 6$, then the incidence graph on P is a strongly regular graph with parameters $(v,k,\lambda,\mu) = (|P|, k^P(k^L-1), k^L-2, 1)$.

For the diameter bound of Moore geometries, see [8, 7, 10, 11] and [5, Section 6.8]

Remark. In the case Theorem 2.6.(iii), the smallest possible value for m is $r+2$ if the minimum valency is at least 2. By the previous lemma, we have $r = 2$ or 4. We treat these cases in the following sections. But it may be possible to give a bound of $r = r(\Gamma)$ of distance-regular graphs satisfying $a_1 = 0$, $c_{r+1} = c_{r+2}$ with r even, by showing the existence of geodetically closed subgraphs of diameter $r+2$, i.e., graphs discussed in the previous lemma.

3 A Refinement of a Theorem of Ray-Chaudhuri and Sprague

In [24], Ray-Chaudhuri and Sprague proved the following theorem in the context of incidence systems.

Theorem 3.1 *Let Γ be a connected bipartite graph with a bipartition $P \cup L$. For some positive integer q, suppose $c_2(x) = 1$, $c_3(x) = c_4(x) = q+1$ for every $x \in P$. Then Γ is* *biregular of valencies* k^P *and* k^L. *If* $k^P > q+1$ *and* $k^L \geq q^2+q+1$, *then* $\Gamma \simeq J_q(d,s,s-1)$, *where* s *and* d *are real numbers defined by*

$$k^L = (q^s - 1)/(q-1), \quad k^P = (q^{d-s+1} - 1)/(q-1).$$

In particular, q *is a power of a prime number and both* s *and* d *are integers.*

The first part of this section is the following: By reviewing the proof of Ray-Chaudhuri and Sprague, we show that we can conclude either $d(\Gamma) \leq 4$ or $\Gamma \simeq J_q(d,s,s-1)$ if we can construct a geodetically closed subgraph of diameter 4 having vertices of valency $q+1$ and that such a subgraph exists if one of the valencies k^P or k^L is at least $3q$. Roughly speaking, we want to decrease the lower bound of the condition on the valencies in the hypothesis from q^2+q+1 to $3q$.

Before we start, we prepare a proposition.

Proposition 3.2 *Let* Γ *be a connected regular graph of valency* k *and diameter* d. *Suppose the distance-2-graph* $\Delta = \Gamma^{(2)}$ *is distance-regular of diameter* $\tilde{d}$. *If each pair of vertices* α, β *at distance three in* Γ *is contained in a shortest circuit of odd length* $2m+1$, *then* $\tilde{d} = m$ *and a connected component of* $\Delta_{\tilde{d}}(\alpha)$ *is a clique of size* k. *Moreover,* $\Delta_{\tilde{d}}(\alpha)$ *is connected if and only if* $d = \tilde{d}$ *and* Γ *is a generalized Odd graph, i.e., a distance-regular graph such that* $a_i = 0$, $i = 1, \ldots, d-1$ *and* $a_d \neq 0$.

Proof. Firstly, we have $a_1 = \cdots = a_{m-1} = 0$, $m \geq 3$. And we have the following.

$$\Delta_1(\alpha) = \Gamma_2(\alpha),\ \Delta_{m-1}(\alpha) \supset \Gamma_3(\alpha),\ \Delta_m(\alpha) \supset \Gamma_1(\alpha).$$

Let $\beta \in \Gamma_1(\alpha)$. Then $\Delta_{m+1}(\alpha) \cap \Delta_1(\beta) = \emptyset$, $\tilde{d} = m$. Moreover,

$$\Gamma_1(\alpha) \setminus \{\beta\} \subset \Delta_1(\beta) \cap \Delta_{\tilde{d}}(\alpha) \subset \Gamma_1(\alpha) \setminus \{\beta\}.$$

Hence $\tilde{a}_{\tilde{d}} = k-1$ and a connected component of $\Delta_{\tilde{d}}(\alpha)$ containing β is a clique of size k.

If $\Delta_{\tilde{d}}(\alpha)$ is connected, as Δ is distance-regular, $\Delta_{\tilde{d}}(\gamma) = \Gamma_1(\gamma)$ is a clique of size k in Δ for every $\gamma \in \Gamma$. Hence Γ is a generalized Odd graph. See [1], [2, Section III.4], and [5, Section 4.2].

In the following we also treat the case when Γ is a k-regular with the same conditions on c_i's as those in Theorem 3.1.

Let q be a positive integer and r a positive even integer. A connected graph Γ is said to be a *$P(r,q)$-graph* if c_i, a_j exist for $1 \leq i \leq r+2$, $1 \leq j \leq r+1$ and they satisfy

$$c_1 = \cdots = c_r = 1,\ a_1 = \cdots = a_{r+1} = 0,\ c_{r+1} = c_{r+2} = q+1.$$

Lemma 3.3 *Let* q *be a positive integer and* r *an even positive integer. The following hold.*

(1) *Let Γ be a connected bipartite graph of diameter at least $r+1$ with a bipartition $P \cup L$. If c_i^P exists for $1 \leq i \leq r+2$, and $c_1^P = \cdots c_r^P = 1$, $c_{r+1}^P = c_{r+2}^P = q+1$, then Γ is a $P(r,q)$-graph.*

(2) *Let Γ be a $P(r,q)$-graph. Then one of the following holds.*

(i) *Γ is a bipartite biregular (possibly regular) graph; or*

(ii) *Γ is a nonbipartite regular graph, i.e., a regular graph containing a circuit of odd length.*

Proof. (1) This follows from Proposition 2.1.(1), (2).
(2) This follows from Proposition 2.3.

Let Γ be a $P(r,q)$-graph of diameter at least $r+1$. According to the previous lemma, there are two possibilities.

(i) Γ is a bipartite graph with a bipartition $P \cup L$ and biregular of valencies k^P and k^L.

(ii) Γ is a nonbipartite graph and regular of valency k. In this case, let $\Gamma = P = L$.

We give a list of known $P(r,q)$-graphs, which is not a polygon. $r = 2$ for the first three examples and $r = 4$ for the rest.

1. $J_q(d,s,s-1)$.

2. O_k, the Odd graph of valency k, (nonbipartite).

3. $2M_7$, the doubled Hoffman-Singleton graph, $(d = 5,\ q = 5)$.

4. ${}^2M_k$, $k = 3,\ 7$, $(d = 6,\ q = 1)$.

5. Foster graph, that is the three fold cover of the incidence graph of $GQ(2,2)$, the generalized quadrangle of order $(2,2)$, $(d = 8,\ q = 1)$.

In this section we study $P(2,q)$-graphs. Let Γ be a $P(2,q)$-graph of diameter at least five.

For $\alpha,\ \beta \in \Gamma$ with $\partial(\alpha,\beta) = 2$ and $\gamma \in C(\alpha,\beta)$, let

$$T(\alpha,\beta) = \Gamma_2(\alpha) \cap \Gamma_2(\beta) \cap \Gamma_3(\gamma).$$

We say Γ satisfies the condition $\#^L$ [resp. $\#^P$], if $\delta,\ \eta \in T(\alpha,\beta)$ implies $\partial(\delta,\eta) \leq 2$ for all $\alpha,\ \beta \in L$ [resp. P] with $\partial(\alpha,\beta) = 2$.

The condition above is called 'Pasch's axiom' in [24].

Lemma 3.4 **(1)** *If $k^L \geq 3q$ or $q = 1$, then Γ satisfies the condition $\#^L$.*

(2) *If $k^P \geq 3q$ or $q = 1$, then Γ satisfies the condition $\#^P$.*

Proof. By symmetry it suffices to prove (1).
Let m_1, $m_2 \in L$ with $\partial(m_1, m_2) = 2$ and $\{x\} = C(m_1, m_2)$. Let $T = T(m_1, m_2)$.
If $l \in T$, then $C(m_2, l) \subset \Gamma_3(m_1)$. Hence

$$|T| = |T(m_1, m_2)| = b_2^L(c_3^L - 1) = (k^L - 1)q.$$

Suppose the condition $\#^L$ fails. Then there exist l, $l' \in T$ with $\partial(l, l') = 4$. Let $\{x_i\} = C(l, m_i)$, $\{x_i'\} = C(l', m_i)$, $i = 1, 2$. Since $c_3 = c_4 = q + 1$, for i, $j = 1, 2$,

$$x_1' \in C(l, l') = C(x_j, l'), \text{ or } \partial(x_i', x_j) = 2.$$

So we have that

$$x_1' \in C(x_2, m_1) \setminus \{x, x_1\},\ x_2' \in C(x_1, m_2) \setminus \{x, x_2\}.$$

Hence $|T \cap \Gamma_4(l)| \leq (q-1)^2$. Similarly, $|T \cap \Gamma_4(l')| \leq (q-1)^2$. In particular, $q \neq 1$. Thus

$$\begin{aligned}
(q+1)^2 &= |\Gamma_2(l) \cap \Gamma_2(l')| \\
&\geq |T \cap \Gamma_2(l) \cap \Gamma_2(l')| + |\{m_1, m_2\}| \\
&\geq |T| + |\{m_1, m_2\}| - |T \cap \Gamma_4(l)| - |T \cap \Gamma_4(l')| \\
&\geq (k^L - 1)q + 2 - 2(q-1)^2.
\end{aligned}$$

So $3q^2 - 2q + 1 \geq (k^L - 1)q$ or $k^L \leq 3q - 1 + \frac{1}{q}$. Since $q \neq 1$, $k^L \leq 3q - 1$, as desired.

For m_1, $m_2 \in \Gamma_2(l)$ with $m_1 \neq m_2$, we write $m_1 \approx m_2$ if $\partial(m_1, m_2) = 2$ and $C(m_1, m_2) \subset \Gamma_3(l)$, or equivalently if $m_2 \in T(l, m_1)$. Since the relation $\approx$ is symmetric, it defines a graph on $\Gamma_2(l)$.

Let $L_1(l, m)$ be a connected component in $\Gamma_2(l)$ containing m with respect to $\approx$. Let

$$L(l, m) = \{l\} \cup L_1(l, m),\ P(l, m) = \bigcup_{n \in L(l,m)} \Gamma(n),\ \Delta(l, m) = P(l, m) \cup L(l, m).$$

Lemma 3.5 *Suppose Γ satisfies the condition $\#^L$. Then for l, $m \in L$ with $\partial(l, m) = 2$, $\Delta = \Delta(l, m)$ is a geodetically closed subgraph of Γ of diameter 4.*

Proof. Since Γ satisfies the condition $\#^L$, we have $\partial(m_1, m_2) \leq 2$, if m_1, $m_2 \in T(l, m)$. Hence we can prove the assertion without difficulty.

Let $D = \{\Delta(l, m) | \partial(l, m) = 2,\ l,\ m \in L\}$.

Corollary 3.6 *If Γ satisfies the condition $\#^L$, then the following hold.*

(1) *$L(l, m)$ is a maximal clique in $\Gamma^{(2)}$.*

(2) *If* $l,\ m \in \Delta_1 \cap \Delta_2 \cap L$*, then* $\Delta_1 = \Delta_2$ *or* $l = m$.

(3) Δ *is a bipartite biregular graph of valencies* $q+1$ *on* $P(l,m)$ *and* k^L *on* $L(l,m)$.

(4) $|L(l,m)| = qk^L + 1$.

(5) $|\{\Delta \in D | l \in \Delta\}| = (k^P - 1)/q$ *for every* $l \in L$.

Let Π be a bipartite graph on $L \cup D$ with adjacency defined as follows: For $l \in L,\ \Delta \in D,\ l \in \Delta$ and the valency of l in Δ is k^L. Note that $k^L > q+1$ as $d(\Gamma) \geq 5$.

Lemma 3.7 *If* Γ *satisfies the condition* $\#^L$*, then* Π *is a* $P(2,q)$*-graph of valencies* $(k^P - 1)/q$ *on* L *and* $qk^L + 1$ *on* D.

Proposition 3.8 *Let* Γ *be a* $P(2,q)$*-graph of diameter at least five satisfying the condition* $\#^L$*. Then one of the following holds.*

(i) $\Gamma \simeq J_q(d,s,s-1)$, *where* $k^L = (q^s - 1)/(q-1)$, $k^P = (q^{d-s+1} - 1)/(q-1)$, *or*

(ii) Γ *is a regular nonbipartite graph of valency* k *and* $\Gamma^{(2)}$ *is isomorphic to a connected component of the distance-2-graph of* $J_q(2s-3, s-2, s-3)$, *where* $k = (q^{s-1} - 1)/(q-1)$. *Moreover, if each pair of vertices of* Γ *at distance three is contained in a shortest circuit of odd length, then* $q = 1$ *and* Γ *is isomorphic to an Odd graph.*

Proof. Firstly, note that $J_q(d,s,s-1) \simeq J_q(d, d-s+1, d-s)$, if we take the dual interchanging P and L.

Suppose Γ is bipartite. Since $d(\Gamma) \geq 5$, $k^P,\ k^L > q+1$. By Theorem 3.1, (i) holds if $k^P \geq q^2 + q + 1$, using the first remark above.

Assume $k^P < q^2 + q + 1$. Since Γ satisfies the condition $\#^L$, Π is a $P(2,q)$-graph of valencies $(k^P - 1)/q$ on L. Since $(k^P - 1)/q < q + 1$, $\partial_\Pi(l,m) \leq 2$ for all $l,\ m \in L$. Hence $\partial_\Gamma(l,m) \leq 2$ for all $l,\ m \in L$, which is not the case.

Suppose Γ is not bipartite. By the previous lemma, Π is a bipartite $P(2,q)$-graph of valencies $(k-1)/q$ on L and $qk+1$ on D.

Suppose $(k-1)/q \leq q+1$. Since $d(\Gamma) \geq 5$, there are vertices $l_0,\ l_1,\ l_2,\ l_3$ such that

$$\partial(l_0, l_1) = \partial(l_1, l_2) = \partial(l_2, l_3) = 2, \quad \partial(l_0, l_2) = 4.$$

Since $|\Pi_3(l_0) \cap \Pi(l_2)| = q+1$, $(k-1)/q = q+1$ and $\Delta(l_2, l_3) \in \Pi_3(l_0) \cap \Pi(l_2)$. So there is a vertex $l \in \Delta(l_2, l_3)$ such that $\partial(l, l_3) = \partial(l_0, l) = 2$. Hence $\partial(l_3, l_0) \leq 4$. In particular $d(\Gamma) = 5$, a_5 exists and $a_5 = 0$. Since Γ is not bipartite, we may assume that $\partial(l_0, l_3) = 3$. Then $|\Gamma_2(l_3) \cap \Gamma_2(l_0)| = 0$. This is a contradiction.

Thus $(k-1)/q > q+1$, $qk+1 > q^2+q+1$. Hence by Theorem 3.1, $\Pi \simeq J_q(d,s,s-1)$, where $qk+1 = (q^s - 1)/(q-1)$, $(k-1)/q = (q^{d-s+1} - 1)/(q-1)$.

Therefore $k = (q^{s-1} - 1)/(q - 1)$ and $d = 2s - 3$. Since $\partial_\Gamma(l, m) = 2$ if and only if $\partial_\Pi(l, m) = 2$, $\Gamma^{(2)}$ is isomorphic to a connected component of the distance-2-graph of Π on L.

If Γ satisfies the additional condition in (ii), we can apply Proposition 2.2. If $q \neq 1$, then $\Gamma^{(2)}$ is a Grassman graph, which is also called a q-analogue of Johnson graph. But in this case it is easy to check that the antipode is connected, while it is not a clique. Hence $q = 1$ and $\Gamma^{(2)} \simeq J(2s - 3, s - 2)$. Thus Γ is an Odd graph.

In the following, we investigate the case when Γ does not satisfy $\#^L$. By symmetry proved in Lemma 3.3, we may assume that Γ does not satisfy $\#^P$ either. Hence by Lemma 3.4, we need only to consider the case k^P, $k^L \leq 3q - 1$.

The key to analize this case is the following proposition proved by Terwilliger. We kept the notations in [30], where M_i is no longer a Moore graph.

Proposition 3.9 ([30]) *Let integers c, p and s all be at least 2. Suppose the vertices of some graph Γ can be partitioned into $s + 1$ disjoint sets $V\Gamma = \bigcup_{i=0}^{s} M_i$, where for any u, $v \in V\Gamma$, $u \in M_i$, $v \in M_j$ and $(u, v) \in E\Gamma$ implies $|i - j| \leq 1$. For $i = 1$ or s, let l_i and L_i denote the minimum and maximum number of vertices in M_{i-1} any vertex in M_i is adjacent to, and for $i = 0$ or $s - 1$, let r_i and R_i denote the minimum and maximum number of vertices in M_{i+1} any vertex in M_i is adjacent to. Also assume*

(i) *$\partial(u, v) = s$ for some $u \in M_0$ and $v \in M_s$,*

(ii) *for integers $0 \leq i$, $j \leq s$ and for any $u \in M_i$ and $v \in M_j$, there are either c or 0 paths of length s connecting them if $|j - i| = s$, and either 0 or 1 paths of length $|j - i|$ connecting them if $1 \leq |j - i| \leq s - 1$, and*

(iii) *for any u, $v \in V\Gamma$ with $u \in M_1$, $v \in M_{s-1}$, and $\partial(u, v) > s - 2$, there are at most p paths $\{u = v_0, v_1, \ldots, v_{s-1}, v_s = v\}$, where either $v_1 \in M_0$ or $v_{s-1} \in M_s$.*

Then

$$\frac{p}{c-1} \geq \frac{r_{s-1}}{R_0 - 1} + \frac{l_1}{L_s - 1}.$$

Proposition 3.10 *Let Γ be a $P(2, q)$-graph of diameter at least five. If c_5^P exists, then c_5 exists, i.e., c_5^L exists and $c_5^P = c_5^L$, $c_5 > q + 1$ and the following hold.*

(1) *If $d(\Gamma) \geq 7$, then $c_5 \geq 2q + 1$.*

(2) *If α, β, $\gamma \in \Gamma$ with $\partial(\alpha, \beta) = 8$, $\partial(\alpha, \gamma) = 3$, $\partial(\gamma, \beta) = 5$, then $k(\gamma) \geq 3q + 2$.*

(3) *For $\alpha \in \Gamma$ let $j = k(\alpha) - c_5$. If $a_4 = 0$, then*

$$k(\alpha) \geq \frac{2q + j + 3 + \sqrt{4jq^2 + (j-1)^2}}{2}.$$

In particular, if $j \geq 4$, then $k(\alpha) \geq 3q + 4$.

Proof. It follows from Proposition 2.1.(2) that c_5 exists.

(1) Let α, $\beta \in \Gamma$ with $\partial(\alpha,\beta) = 7$. Let

$$M_i = \Gamma_{2+i}(\alpha) \cap \Gamma_{5-i}(\beta), \ i = 0,\ 1,\ 2,\ 3.$$

Apply Proposition 3.9.

(2) Since $d \geq 8$, we can apply (1). We have

$$k(\gamma) \geq c_3(\alpha,\gamma) + c_5(\beta,\gamma) \geq 3q + 2.$$

(3) Let $\alpha \in \Gamma$ and $M_i = \Gamma_{i+2}(\alpha)$, $i = 0,\ 1,\ 2,\ 3$. Apply Proposition 3.8.

We now summarize our results in this section, from which we have Theorem 1.3 as a corollary.

Theorem 3.11 *Let Γ be a $P(2,q)$-graph of diameter at least five. Suppose c_5 exists. Then Γ is a bipartite biregular graph of valencies k^P and k^L, or a regular graph of valency $k = k^P = k^L$ and one of the following holds.*

(i) $\Gamma \simeq J_q(d,s,s-1)$, *where* $k^L = (q^s-1)/(q-1)$, $k^P = (q^{d-s+1}-1)/(q-1)$,

(ii) *Γ is a regular nonbipartite graph of valency k and the distance-2-graph $\Gamma^{(2)}$ is isomorphic to a connected component of the distance-2-graph of $J_q(2s-3,s-2,s-3)$, where $k = (q^{s-1}-1)/(q-1)$. Moreover, if each pair of vertices of Γ at distance three is contained in a shortest circuit of odd length, then $q = 1$ and Γ is isomorphic to an Odd graph; or*

(iii) *$d(\Gamma) \leq 7$ and k^P, $k^L \leq 3q-1$, $q \neq 1$. Moreover if $a_4 = 0$, then Γ is bipartite and $k^P - c_5$, $k^L - c_5 \leq 3$. In particular, if Γ is not bipartite and a_4 exists, then $d(\Gamma) \leq 6$.*

Corollary 3.12 *Let Γ be a distance-regular graph of valency k. Suppose $c_2 = 1$, $c_3 = c_4 = q+1$ and $a_1 = a_2 = a_3 = 0$ for some positive integer q. Then one of the following holds.*

(i) $\Gamma \simeq J_q(2s-1,s-2,s-3)$, *where* $k = (q^s-1)/(q-1)$.

(ii) $\Gamma \simeq O_k$, *an Odd graph of valency k; or*

(iii) $d(\Gamma) \leq 7$, *and the equality holds only if Γ is bipartite.*

Koolen [20] conjectured the following:

> If Γ is a distance-biregular graph of diameter at least 5 such that c_i exists for all i, and $c_2 = 1$, $c_3 = c_4 > 2$, then $\Gamma \simeq J_q(d,s,s-1)$.

Our results asserts that $d(\Gamma) \leq 7$ and the parameters are restricted very much. It is known that if $d(\Gamma) = 5$ or 7, then Γ is distance-regular, under the assumption of the conjecture above. See [9, 20].

We also note that for $d(\Gamma) = 5$, the doubled Moore graph satisfy the hypothesis with $c_5 = q+2$. Moreover if it's valency is not 3, say 7, then it does not come from $J_q(d, s, s-1)$. So this gives a counter example to the conjecture above.

4 $P(r, 1)$-graphs

According to the remark following Lemma 3.3, a $P(r, 1)$-graph is a connected graph Γ, which is either a bipartite biregular graph with a bipartition $P \cup L$ or a nonbipartite regular graph such that

$$c_1 = \cdots = c_r = 1, \ a_1 = \cdots = a_{r+1} = 0, \ c_{r+1} = c_{r+2} = 2,$$

where r is an even positive integer. In this section we study $P(r, 1)$-graphs and we show the following when $r = 4$. We do not know any $P(r, 1)$-graphs with $r > 4$.

Theorem 4.1 *Let Γ be a $P(4, 1)$-graph of diameter at least four and α, $\gamma \in \Gamma$ with $\partial(\alpha, \gamma) = 4$. Then there is a geodetically closed subgraph Δ containing α, γ isomorphic to ${}^2M_{k(\alpha)}$. Here $k(\alpha)$ denotes the valency of α in Γ. In particular, $k(\alpha) \in \{2, 3, 7, 57\}$.*

Let Γ be a $P(r, 1)$-graph with $r \geq 4$.

Fix a vertex $\alpha \in \Gamma$. For γ, $\delta \in \Gamma_r(\alpha)$, we write $\gamma \approx \delta$ if $\partial(\gamma, \delta) = 2$ and $C(\gamma, \delta) \subset \Gamma_{r+1}(\alpha)$. For $\gamma \in \Gamma_r(\alpha)$, let $C = C_\gamma$ be the connected component in $\Gamma_r(\alpha)$ containing γ with respect to the relation $\approx$. Let $\Pi = \Pi_\gamma$ be a graph on C_γ defined by the relation $\approx$. For γ, $\delta \in \Gamma$ with $\partial(\gamma, \delta) = r$, and $0 \leq i \leq r$, let

$$\{g_i(\gamma, \delta)\} = \Gamma_{r-i}(\gamma) \cap \Gamma_i(\delta).$$

For $\delta \in \Gamma_r(\alpha)$, let

$$\alpha(\delta) = g_1(\delta, \alpha), \ \beta(\delta) = g_2(\delta, \alpha), \ \text{ and } \ \gamma(\delta) = g_4(\delta, \alpha).$$

Firstly we note that the intersection diagram with respect to x, l with $\partial(x, l) = 1$ has the following shape, where $D^i_j = \Gamma_i(x) \cap \Gamma_j(l)$. See the properties (a) $\sim$ (e) below.

$$\begin{tikzcd}[arrows=-]
\{x\} = D^0_1 \arrow[r] \arrow[dd] & \cdots\cdots \arrow[r] & D^{r-1}_r \arrow[r] & D^r_{r+1} \arrow[r] \arrow[dd] & D^{r+1}_{r+2} \arrow[r] \arrow[dr] & {} \\
 & & & & {} & D^{r+2}_{r+2} \arrow[r] \arrow[dl, ->] \arrow[ur] \arrow[dr] & {} \\
\{l\} = D^1_0 \arrow[r] & \cdots\cdots \arrow[r] & D^r_{r-1} \arrow[r] & D^{r+1}_r \arrow[r] & D^{r+2}_{r+1} \arrow[r] & {} & {}
\end{tikzcd}$$

Figure 2.

(a) $D_i^i = \emptyset$, for $1 \leq i \leq r+1$.

(b) For $y \in D_i^{i+1}$, $z \in D_{i+1}^i$, $e(y, D_{i-1}^i) = e(z, D_i^{i-1}) = 1$, $1 \leq i \leq r$.

(c) For $y \in D_{r+1}^{r+2}$, $z \in D_{r+2}^{r+1}$, $e(y, D_r^{r+1}) = e(z, D_{r+1}^r) = 2$.

(d) For $y \in D_r^{r+1}$, $z \in D_{r+1}^r$, $e(y, D_{r+1}^r) = e(z, D_r^{r+1}) = 1$.

(e) $e(D_i^{i+1}, D_{i+1}^i) = 0$, $1 \leq i \leq r-1$ and $i = r+1$.

The following two lemmas are related to circuit chasing technique. See [4, 13, 14] and [5, Section 5.10].

Lemma 4.2 *Let $x_0 \sim x_1 \sim \cdots \sim x_{2r+2t} = x_0$ be a circuit of length $2r+2t$. i.e., a closed path and $x_{i-1} \neq x_{i+1}$, $i = 1, \ldots, 2r+2t-1$ and $x_{2r+2t-1} \neq x_1$. Suppose*

$$x_r,\ x_{r+2}, \ldots,\ x_{r+2t} \in \Gamma_r(x_0),\ x_{r+1},\ x_{r+3},\ \ldots,\ x_{r+2t-1} \in \Gamma_{r+1}(x_0).$$

Set $D_j^i = \Gamma_i(x_0) \cap \Gamma_j(x_1)$. Then the following hold.

(1) *$t \geq 1$ and $x_r \in D_{r-1}^r$, $x_{r+1} \in D_r^{r+1}$, $x_{r+2} \in D_{r+1}^r$.*

(2) *If $t \geq 2$, then $x_{r+3} \in D_{r+2}^{r+1}$ and $x_{r+4} \in D_{r+1}^r$.*

(3) *If $t = 2$, then the mutual distance of the vertices in the circuit is uniquely determined. In particular,*

$$\partial(x_2, x_{r+2}) = \partial(x_2, x_{r+4}) = r,\ \partial(x_2, x_{r+5}) = r+1.$$

(4) *If $t = 3$, then $x_{r+5} \in D_{r+2}^{r+1}$, $x_{r+6} \in D_{r+1}^r$ and*

$$\partial(x_2, x_{r+4}) = \partial(x_2, x_{r+6}) = \partial(x_4, x_{r+6}) = r,\ \partial(x_4, x_{r+5}) = \partial(x_4, x_{r+7}) = r+1.$$

Proof. In the following, we use (a) $\sim$ (e) to determine the locations of x_j's in the diagram with respect to an edge $x_{i-1} \sim x_i$, using the information on the distances from x_{i-1}.

(1) Since $x_{i-1} \neq x_{i+1}$, for all i, and $c_1 = \cdots = c_r = 1$, $t \geq 1$. It is clear that $x_r \in D_{r-1}^r$. Since $x_{r+1} \in \Gamma_{r+1}(x_0) \cap \Gamma(x_r)$, $x_{r+1} \in D_r^{r+1}$. $x_r \neq x_{r+2} \in \Gamma_r(x_0) \cap \Gamma(x_{r+1})$ implies that $x_{r+2} \in D_{r+1}^r$.

(2) Since $x_{r+2} \in D_{r+1}^r$ and $e(x_{r+2}, D_r^{r+1}) = 1$ with $x_{r+1} \in D_r^{r+1} \cap \Gamma(x_{r+2})$, $x_{r+3} \in D_{r+2}^{r+1}$, $x_{r+4} \in D_{r+1}^r$.

(3) It is easy to determine the mutual distances as follows.

	x_r	x_{r+1}	x_{r+2}	x_{r+3}	x_{r+4}	x_{r+5}
x_0	r	$r+1$	r	$r+1$	r	$r-1$
x_1	$r-1$	r	$r+1$	$r+2$	$r+1$	r
x_2	$r-2$	$r-1$	r	$r+1$	r	$r+1$

Now the distance pattern with respect to x_2 is the same as that with respect to x_0, the mutual distance of the vertices in the circuit is uniquely determined and the assertion follows.

(4) We do the same as in (3).

	x_r	x_{r+1}	x_{r+2}	x_{r+3}	x_{r+4}	x_{r+5}	x_{r+6}	x_{r+7}	x_{r+8}	x_{r+9}
x_0	r	$r+1$	r	$r+1$	r	$r+1$	r	$r-1$	$r-2$	$r-3$
x_1	$r-1$	r	$r+1$	$r+2$	$r+1$	$r+2$	$r+1$	r	$r-1$	$r-2$
x_2	$r-2$	$r-1$	r	$r+1$	r	$r+1$	r	$r+1$	r	$r-1$
x_3	$r-3$	$r-2$	$r-1$	r	$r+1$	$r+2$	$r+1$	$r+2$	$r+1$	r
x_4	$r-4$	$r-3$	$r-2$	$r-1$	r	$r+1$	r	$r+1$	r	$r+1$

Note that since $x_{r+7} \in D_r^{r-1}$, x_{r+5} cannot be in D_r^{r+1}.

Lemma 4.3 *Let $y_0 \sim y_1 \sim y_2 \sim y_3 \sim y_4$ be a path of length four such that $y_{i-1} \neq y_{i+1}$, $i = 1, \dots, 3$. Suppose y_0, $y_4 \in \Gamma_r(\alpha)$. Then one of the following holds.*

(i) $y_2 \in \Gamma_{r-2}(\alpha)$,

(ii) $y_1 \in \Gamma_{r-1}(\alpha)$ *or* $y_3 \in \Gamma_{r-1}(\alpha)$ *and* $\alpha(y_0) \neq \alpha(y_4)$,

(iii) y_1, $y_3 \in \Gamma_{r+1}(\alpha)$, $y_2 \in \Gamma_r(\alpha)$ *and* $\alpha(y_0) \neq \alpha(y_4)$,

(iv) $y_2 \in \Gamma_{r+2}(\alpha)$ *and* $\alpha(y_0) = \alpha(y_4)$, *while* $\beta(y_0) \neq \beta(y_4)$, *or*

(v) $y_2 \in \Gamma_{r+2}(\alpha)$ *and* $\alpha(y_0) \neq \alpha(y_4)$, $\partial(\beta(y_0), y_4) = r + 2$.

By Lemma 4.2 and 4.3, we can prove the following concerning the connected component in $\Gamma_r(\alpha)$ with respect to $\approx$.

Lemma 4.4 *Let $\{\alpha_1, \dots, \alpha_{k(\alpha)}\} = \Gamma(\alpha)$, $\gamma \in \Gamma_r(\alpha)$, $C = C_\gamma$. Let $S_i = \{\delta \in C | \alpha(\delta) = \alpha_i\}$. Then the following hold.*

(1) *For $\delta \in S_i$, $|\Pi(\delta) \cap S_j| = 1 - \delta_{i,j}$ and $S_i \subset \Gamma_{r-2}(\beta(\delta))$. In particular, Π is a $k(\alpha)$-partite $(k(\alpha) - 1)$-regular graph.*

(2) *Let $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3$ be a path in Π. If $\alpha(\delta_0) \neq \alpha(\delta_3)$, then there exists $\delta_4 \in \Pi(\delta_3)$, $\delta_5 \in \Pi(\delta_4)$ such that $\gamma(\delta_0) = \gamma(\delta_5)$.*

If $r = 4$, $\gamma(\delta) = \delta$ for every $\delta \in \Pi$. So by Lemma 4.4, we have the following.

Lemma 4.5 *If $r = 4$, then the following holds.*

(1) *If $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3$ and $\alpha(\delta_0) \neq \alpha(\delta_3)$, then there exists δ_4 such that $\delta_0 \approx \delta_4 \approx \delta_3$.*

(2) *If* $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3$ *and* $\alpha(\delta_0) = \alpha(\delta_3)$, *then* $\beta(\delta_0) = \beta(\delta_3)$.

(3) *If* $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3 \approx \delta_4$ *with* $\alpha(\delta_0) = \alpha(\delta_3)$, $\alpha(\delta_1) = \alpha(\delta_4)$, *then there exists* δ_5 *such that* $\delta_0 \approx \delta_5 \approx \delta_4$.

(4) $d(\Pi) \leq 3$ *and if* $\partial_\Pi(\delta, \delta') = 3$, *then* $\beta(\delta) = \beta(\delta')$.

Proof. (1) Since $\gamma(\delta) = \delta$ for every $\delta \in \Pi$, (1) is a direct consequence of Lemma 4.4.(2).

(2) This follows from Lemma 4.4.(1).

(3) By (2), $\beta(\delta_0) = \beta(\delta_3) \neq \beta(\delta_1) = \beta(\delta_4)$. Now δ_3, $\beta(\delta_1) \in \Gamma_4(\delta_0)$, and there is a path of length 4,

$$y_0 = \delta_3 \sim y_1 \sim y_2 = \delta_4 \sim y_3 \sim y_4 = \beta(\delta_1),$$

where $y_1 \in C(\delta_3, \delta_4)$, $y_3 = g_1(\alpha, \delta_4)$.

It is easy to check that y_1, $y_3 \in \Gamma_5(\delta_0)$ and that $g_1(\delta_3, \delta_0) \neq g_1(\beta(\delta_1), \delta_0)$. Hence by Lemma 4.3.(iii) or (v) occurs.

If (v) occurs, $\partial(\beta(\delta_0), \delta_4) = 6$, which is not the case. Hence $\partial(\delta_0, \delta_4) = 4$.

Let $\delta_0 = z_0 \sim z_1 \sim z_2 \sim z_3 \sim z_4 = \delta_4$ be a path connecting δ_0 and δ_4. Then by Lemma 4.3, we have (iii) as $\partial(\beta(\delta_0), \delta_4) = 4$. Hence we can set $z_2 = \delta_5$.

(4) This follows from (1), (2) and (3).

Proof of Theorem 4.1. Let $r = 4$ and

$$\begin{aligned} L(\alpha,\gamma) &= \{\alpha\} \cup \bigcup_{\delta \in C_\gamma} (\Gamma_2(\alpha) \cap \Gamma_2(\delta)) \cup C_\gamma, \\ P(\alpha,\gamma) &= \bigcup_{\delta \in L(\alpha,\gamma)} \Gamma_1(\delta), \\ \Delta &= \Delta(\alpha,\gamma) = P(\alpha,\gamma) \cup L(\alpha,\gamma) \end{aligned}$$

In this definition we also write $P(\Delta) = P(\alpha,\gamma)$, and $L(\Delta) = L(\alpha,\gamma)$.

We shall show in the sequel that Δ is a geodetically closed subgraph isomorphic to ${}^2M_{k(\alpha)}$.

Let $\gamma = \gamma_1$ and $\{\gamma_2, \ldots, \gamma_{k(\alpha)}\} = \Pi(\gamma)$. Thanks to Lemma 4.4,

$$L(\Delta) = \{\alpha\} \cup \{\beta(\gamma_1), \ldots, \beta(\gamma_{k(\alpha)})\} \cup C_\gamma.$$

By Lemma 4.5, the distance-2-graph induced on $L(\Delta)$ is of diameter 2 and geodetically closed.

If $k(\alpha) = 2$, there is nothing to prove. Assume $k(\alpha) > 2$.

$\partial(\beta(\gamma), \gamma_2) = 4$ and

$$\Pi(\gamma_2) \setminus \{\gamma_1\} = \{\delta_1, \ldots, \delta_{k(\alpha)-1}\} \subset \Gamma_4(\beta(\gamma)),$$

there is a vertex $\delta_i' \in \Pi(\delta_i) \cap \Gamma_2(\beta(\gamma))$ for each i. Since the girth of Γ is 10, we can conclude that the valenncy of $\beta(\gamma)$ in the distance-2-graph induced on $L(\Delta)$ equals $k(\alpha)$. By Lemma 4.5, this means that the valency of vertex in $P(\Delta)$ is 2.

Now we can conclude that Δ is geodetically closed subgraph of Γ isomotphic to ${}^2M_{k(\alpha)}$ easily.

This completes the proof of Theorem 4.1.

We remark that in the final step, we can also apply [5, Theorem 1.17.1] to determine the regularity of the distance-2-graph induced on $L(\Delta)$. See the proof of [5, Proposition 4.3.11].

5 Proof of Theorem 1.5

In this section, we give a proof of Theorem 1.5. We can follow the proof in the previous section step by step, replacing each path of length 2 by a path of length 3.

Let Γ be a graph satisfying the hypothesis in Theorem 1.5.

Fix a vertex $\alpha \in \Gamma$. For γ, $\delta \in \Gamma_r(\alpha)$, we write $\gamma \approx \delta$ if $\partial(\gamma,\delta) = 3$. Then $C(\gamma,\delta) \cup C(\delta,\gamma) \subset \Gamma_{r+1}(\alpha)$. For $\gamma \in \Gamma_r(\alpha)$, let $C = C_\gamma$ be the connected component in $\Gamma_r(\alpha)$ containing γ with respect to the relation $\approx$. Let $\Pi = \Pi_\gamma$ be a graph on C_γ defined by the relation $\approx$. Hence C is a connected component of the distance-3-graph of Γ induced on the set $\Gamma_r(\alpha)$.

For γ, $\delta \in \Gamma$ with $\partial(\gamma,\delta) = r$, and $0 \le i \le r$, let

$$\{g_i(\gamma,\delta)\} = \Gamma_{r-i}(\gamma) \cap \Gamma_i(\delta).$$

For $\delta \in \Gamma_r(\alpha)$, let

$$\alpha(\delta) = g_1(\delta,\alpha),\ \alpha'(\delta) = g_2(\delta,\alpha),\ \beta(\delta) = g_3(\delta,\alpha),\ \text{ and }\ \gamma(\delta) = g_6(\delta,\alpha).$$

Firstly we note that the intersection diagram with respect to x, y with $\partial(x,y) = 1$ has the following shape, where $D_j^i = \Gamma_i(x) \cap \Gamma_j(y)$. See the properties (a) $\sim$ (g) below.

$$\begin{array}{cccccccccc}
\{x\} = D_1^0 & - & \cdots\cdots & - & D_r^{r-1} & - & D_{r+1}^{r} & - & D_{r+2}^{r+1} & - & D_{r+3}^{r+2} & - \\
| & & & & & & \diagdown & & & & \diagdown & \\
| & & & & & & & D_{r+1}^{r+1} & - & D_{r+2}^{r+2} & - & \\
| & & & & & & \diagup & & & & \diagup & \\
\{y\} = D_0^1 & - & \cdots\cdots & - & D_{r-1}^{r} & - & D_r^{r+1} & - & D_{r+1}^{r+2} & - & D_{r+2}^{r+3} & -
\end{array}$$

Figure 3.

(a) $D_i^i = \emptyset$, for $1 \le i \le r$.

(b) For $y \in D_i^{i+1}$, $z \in D_{i+1}^i$, $e(y, D_{i-1}^i) = e(z, D_i^{i-1}) = 1$, $1 \le i \le r+2$.

(c) For $y \in D_i^{i+1}$, $z \in D_{i+1}^i$, $e(y, D_i^{i+1}) = e(z, D_{i+1}^i) = 0$, $1 \le i \le r$ and $e(y, D_i^{i+1}) = e(z, D_{i+1}^i) = 1$, $i = r+1$, $r+2$.

(d) For $y \in D_{r+1}^{r+1}$, $e(y, D_r^{r+1}) = e(y, D_{r+1}^r) = 1$ and $e(y, D_{r+1}^{r+1}) = 0$.

(e) For $y \in D_r^{r+1}$, $z \in D_{r+1}^r$, $e(y, D_{r+1}^{r+1}) = e(z, D_{r+1}^{r+1}) = 1$.

(f) For $y \in D_{r+2}^{r+2}$, $e(y, D_{r+1}^{r+1}) = e(y, D_{r+2}^{r+2}) = 1$.

(g) $e(D_i^{i+1}, D_{i+1}^i) = 0$, $1 \le i \le r+2$.

We again apply circuit chasing technique.

Lemma 5.1 *Let $x_0 \sim x_1 \sim \cdots \sim x_{2r+3t} = x_0$ be a circuit of length $2r+3t$. i.e., a closed path and $x_{i-1} \ne x_{i+1}$, $i = 1, \ldots, 2r+3t-1$ and $x_{2r+3t-1} \ne x_1$. Suppose*

$$x_r,\ x_{r+3}, \ldots,\ x_{r+3t} \in \Gamma_r(x_0),\ x_{r+1},\ x_{r+2},\ x_{r+4},\ x_{r+5},\ \ldots,\ x_{x+3t-2},\ x_{r+3t-1} \in \Gamma_{r+1}(x_0).$$

Set $D_j^i = \Gamma_i(x_0) \cap \Gamma_j(x_1)$. Then the following hold.

(1) *$t \ge 1$ and $x_r \in D_{r-1}^r$, $x_{r+1} \in D_r^{r+1}$, $x_{r+2} \in D_{r+1}^{r+1}$ and $x_{r+3} \in D_{r+1}^r$.*

(2) *If $t \ge 2$, then x_{r+4}, $x_{r+5} \in D_{r+2}^{r+1}$ and $x_{r+6} \in D_{r+1}^r$.*

(3) *If $t = 2$, then the mutual distance of the vertices in the circuit is uniquely determined. In particular, $r \equiv 0 \pmod 3$, and*

$$\partial(x_3, x_{r+3}) = \partial(x_3, x_{r+6}) = r,\ \partial(x_3, x_{r+7}) = r+1.$$

(4) *Suppose $r \ge 6$. If $t = 3$, then x_{r+7}, $x_{r+8} \in D_{r+2}^{r+1}$, $x_{r+9} \in D_{r+1}^r$ and*

$$\partial(x_3, x_{r+6}) = \partial(x_3, x_{r+9}) = \partial(x_6, x_{r+9}) = r,\ \partial(x_6, x_{r+8}) = \partial(x_6, x_{r+10}) = r+1.$$

Lemma 5.2 *Let $y_0 \sim y_1 \sim y_2 \sim y_3 \sim y_4 \sim y_5 \sim y_6$ be a path of length 6 such that $y_{i-1} \ne y_{i+1}$, $i = 1, \ldots, 5$. Suppose y_0, $y_6 \in \Gamma_r(\alpha)$. Then one of the following holds.*

(i) *$y_3 \in \Gamma_{r-3}(\alpha)$,*

(ii) *y_1, y_2, y_4, $y_5 \in \Gamma_{r+1}(\alpha)$, $y_3 \in \Gamma_r(\alpha)$ and $\alpha(y_0) \ne \alpha(y_6)$,*

(iii) *$y_3 \in \Gamma_{r+2}(\alpha)$ and $y_5 \in \Gamma_{r+1}(\alpha) \cap \Gamma_{r+1}(\alpha(y_0))$, while $\partial(\beta(y_0), y_5) \ge r+1$.*

Lemma 5.3 *Let $\{\alpha_1, \ldots, \alpha_k\} = \Gamma(\alpha)$, $\gamma \in \Gamma_r(\alpha)$, $C = C_\gamma$. Let $S_i = \{\delta \in C | \alpha(\delta) = \alpha_i\}$. Then the following hold.*

(1) *For $\delta \in S_i$, $|\Pi(\delta) \cap S_j| = 1 - \delta_{i,j}$ and $S_i \subset \Gamma_{r-3}(\beta(\delta))$. In particular, Π is a k-partite $(k-1)$-regular graph.*

(2) *Let $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3$ be a path in Π. If $\alpha(\delta_0) \neq \alpha(\delta_3)$, then there exists $\delta_4 \in \Pi(\delta_3)$, $\delta_5 \in \Pi(\delta_4)$ such that $\gamma(\delta_0) = \gamma(\delta_5)$.*

Lemma 5.4 *If $r = 6$, then the following holds.*

(1) *If $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3$ and $\alpha(\delta_0) \neq \alpha(\delta_3)$, then there exists δ_4 such that $\delta_0 \approx \delta_4 \approx \delta_3$.*

(2) *If $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3$ and $\alpha(\delta_0) = \alpha(\delta_3)$, then $\beta(\delta_0) = \beta(\delta_3)$.*

(3) *If $\delta_0 \approx \delta_1 \approx \delta_2 \approx \delta_3 \approx \delta_4$ with $\alpha(\delta_0) = \alpha(\delta_3)$, $\alpha(\delta_1) = \alpha(\delta_4)$, then there exists δ_5 such that $\delta_0 \approx \delta_5 \approx \delta_4$.*

(4) *$d(\Pi) \leq 3$ and if $\partial_\Pi(\delta, \delta') = 3$, then $\beta(\delta) = \beta(\delta')$.*

Proof of Theorem 1.5. Suppose $r = 3$. Let

$$\begin{aligned} L(\alpha,\gamma) &= \{\alpha\} \cup C_\gamma, \\ P(\alpha,\gamma) &= \bigcup_{\delta \in L(\alpha,\gamma)} \Gamma_1(\delta), \\ \Delta &= \Delta(\alpha,\gamma) \;=\; P(\alpha,\gamma) \cup L(\alpha,\gamma) \end{aligned}$$

In this definition we also write $P(\Delta) = P(\alpha,\gamma)$, and $L(\Delta) = L(\alpha,\gamma)$. Clearly $L(\Delta)$ is a maximal clique in the distance-3-graph of Γ, and the assertion follows easily from Lemma 5.3.

Let $r = 6$ and

$$\begin{aligned} L(\alpha,\gamma) &= \{\alpha\} \cup \bigcup_{\delta \in C_\gamma} (\Gamma_3(\alpha) \cap \Gamma_3(\delta)) \cup C_\gamma, \\ P(\alpha,\gamma) &= \bigcup_{\delta \in L(\alpha,\gamma)} \Gamma_1(\delta), \\ \Delta &= \Delta(\alpha,\gamma) \;=\; P(\alpha,\gamma) \cup L(\alpha,\gamma) \end{aligned}$$

In this definition we also write $P(\Delta) = P(\alpha,\gamma)$, and $L(\Delta) = L(\alpha,\gamma)$.

We shall show in the sequel that Δ is a geodetically closed subgraph isomorphic to ${}^3M_{k(\alpha)}$.

Let $\gamma = \gamma_1$ and $\{\gamma_2, \ldots, \gamma_k\} = \Pi(\gamma)$. Thanks to Lemma 4.4,

$$L(\Delta) = \{\alpha\} \cup \{\beta(\gamma_1), \ldots, \beta(\gamma_k)\} \cup C_\gamma.$$

By Lemma 5.4, the distance-3-graph induced on $L(\Delta)$ is of diameter 2 and geodetically closed.

$\partial(\beta(\gamma), \gamma_2) = 6$ and

$$\Pi(\gamma_2) \setminus \{\gamma_1\} = \{\delta_1, \ldots, \delta_{k-1}\} \subset \Gamma_6(\beta(\gamma)),$$

there is a vertex $\delta_i' \in \Pi(\delta_i) \cap \Gamma_3(\beta(\gamma))$ for each i. Since the girth of Γ is 15, we can conclude that the valenncy of $\beta(\gamma)$ in the distance-3-graph induced on $L(\Delta)$ equals k. By Lemma 5.4, this means that the valency of vertex in $P(\Delta)$ is 2.

Now we can conclude that Δ is geodetically closed easily.

This completes the proof of Theorem 1.5.

6 Concluding Remarks

It may be too optimistic to expect a classification of $P(r,q)$-graphs or the graphs similar to those discussed in the previous section in the near future. But we believe that the investigation of such graphs plays a key role to give an absolute bound of the girth of distance-biregular graphs or distance-regular graphs.

We list several problems, which we want to see solved.

1. Study geodetically closed subgraphs of distance-regular graphs and prove results corresponding to Proposition 2.3 and Theorem 2.6, especially when $a_1 \neq 0$. See [20].

2. Classify $P(r,q)$-graphs.

   a) For $r = 2$, it may be possible to improve Lemma 3.4 to have $2q$ as the lower bound. Then we have $d \leq 5$, by Proposition 3.10.

   b) For $q = 1$, the classification implies a classification of distance-biregular graphs with vertices of valency three, [26]. Hence we can obtain an absolute diameter bound of distance-regular graphs of order $(s,2)$, i.e., those with $\Gamma(x) \simeq 3 \cdot K_s$. See [17, 3, 15, 31].

3. Let Γ be a bipartite biregular graph with a bipartition $P \cup L$, or a regular graph with $\Gamma = P = L$. For a positive integer q and a positive odd integer r, we call Γ a $P(r,q)$-graph, if it is a connected graph such that

$$c_1^P = \cdots = c_r^P = 1,\ a_1 = \cdots = a_{r+1} = 0,\ c_{r+1}^P = q+1 \ \text{ and } \ c_{r+1}^L = c_{r+2}^P.$$

   Classify them. If $q = 1$, then Γ is a thin generalized polygon by a result in [26].

4. Study a distance-regular graphs Γ with $r = r(\Gamma)$, $c_{r+1} = c_{r+2} = 1$, and clarify the correspondence with $P(r,q)$-graphs. In particular, show $r \leq 6$ in Theorem 1.5.

5. Let Γ be a connected graph of diameter d. For a subset $I \subset \{1,\ldots,d\}$, let $\Gamma^{(I)}$ denote the distance-I-graph, i.e., $V(\Gamma^{(I)}) = V(\Gamma)$, and α, β are adjacent in $\Gamma^{(I)}$ if and only if $\partial(\alpha,\beta) \in I$. Study Γ such that at least one of the connected components of $\Gamma^{(I)}$ is distance-regular of diameter at least three. To start with, assume $\Gamma^{(I)}$

is connected. It is not hard to determine parametrical conditions if Γ itself is a diatance-regular graph. In particular, classify distance-regular graphs Γ such that $\Gamma^{(2)}$ is distance-regular of diameter $d(\Gamma) \neq d(\Gamma^{(2)}) \geq 3$. See Proposition 3.2 and [27, 29].

6. Give a geometrical classification of Moore graphs. One of the reasons, we could not obtain the results for $P(r,1)$-graphs with $r \geq 6$, is a lack of such classification. We believe that this is one of the keys when we develope structure theories of distance-regular graphs just as the group theoretical proof of Burnside's $p^a q^b$ theorem gave a breakthrough to the classification of finite simple groups.

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